

**AH algebras with Ideal Property,
Exponential Rank, Reduction and
Classification**

Guihua Gong

University of Puerto Rico

(Part of this talk is based on joint work with
C. Jiang, L. Li and C. Pasnicu)

Three classes of C^* -algebras:

I. C^* -algebras of real rank zero (Brown-Pedersen)

A is called of real rank zero, if for each $x \in \mathcal{A}_{s,a}$ and $\varepsilon > 0$, $\exists y \in \mathcal{A}_{s,a}$ with finite spectrum such that $\|x - y\| < \varepsilon$. In other words, there are finite many mutually orthogonal projections p_1, p_2, \dots, p_n and real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$\|x - \sum_{i=1}^n \lambda_i p_i\| < \varepsilon$$

II. Simple C^* -algebras: C^* -algebra \mathcal{A} without any nontrivial closed two sided ideal.

III. C^* -algebra with ideal property (Elliott), Any non-trivial closed two sided ideal $I \subset \mathcal{A}$ is generated by the projections inside the ideal.

Obviously Class II \subset Class III

Also Class I \subset Class III

(proof: Any ideal of real rank zero C^* -algebra is also of real rank zero. So in the approximation $\|x - \sum_{i=1}^n \lambda_i p_i\| < \varepsilon$, if $x \in I \subset \mathcal{A}$, then p_i can be chosen to be in I .)

For purely infinite algebras, Class I = Class III (Classification : Rørdam, Elliott-R, Kerchberg, Phillips,...)

Focus on stably finite C^* -algebras.

Last twenty years: There are many significant classification results:

For Class I: Elliott, Su, E-G, Lin, E-G-Lin-Pasnicu, Dadarlat-G, Eilers, D-Loring, D-G.

For Class II: Elliott, Li, Jiang-Su, Thomsen, G, E-G-L. Lin, Niu, Winter and Lin-Niu

Pasnicu studied Class III intensively. But only recently classification becomes possible.

An AH-algebra \mathcal{A} is an inductive limit

$$\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \dots \longrightarrow \mathcal{A}_n \xrightarrow{\phi_n} \dots \longrightarrow \mathcal{A},$$

where $\mathcal{A}_n = \bigoplus_{i=1}^{k_n} A_n^i = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i}))$, $X_{n,i}$ are compact metric spaces, and $[n, i]$ are positive integers.

Notation: $\phi_{n,m} : \mathcal{A}_n \rightarrow \mathcal{A}_m$,

$$\phi_{n,m} = \phi_{m-1} \circ \phi_{m-2} \circ \dots \circ \phi_n : \mathcal{A}_n \rightarrow \mathcal{A}_m.$$

Elliott-G-Li: One can replace $X_{n,i}$ by connected simplicial complex and $\phi_{n,m}$ by injective homomorphisms.

No dimension growth: $\sup \dim(X_{n,i}) < \infty$.

Slow dimension growth (Blackadar-Dadarlat-Rordam):

$\forall n, \exists d_n$ such that

$$\lim_{m \rightarrow \infty} \min_{\substack{\dim(X_{m,j}) > d_n \\ \phi_{n,m}^{i,j} \neq 0}} \frac{\text{rank}(\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i}))}{\dim(X_{m,j})} = +\infty$$

Very slow dimension growth: $\forall n, \exists d_n$ such that

$$\lim_{m \rightarrow \infty} \min_{\substack{\dim(X_{m,j}) > d_n \\ \phi_{n,m}^{i,j} \neq 0}} \frac{\text{rank}(\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i}))}{(\dim(X_{m,j}))^3} = +\infty$$

In general, a homomorphism $\phi : M_n(C(X)) \rightarrow M_m(C(Y))$

can be written as

$$\phi(f)(y) = u_y \left(\begin{array}{ccccccc} f(x_1(y)) & & & & & & \\ & f(x_2(y)) & & & & & \\ & & \ddots & & & & \\ & & & f(x_\bullet(y)) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array} \right) u_y^*$$

Denote: $Sp\phi|_y = \{x_1(y), x_2(y), \dots, x_\bullet(y)\} \subset X$ counting multiplicity.

For X metric space, $F \subset X$ a closed subset, Denote

$$B\varepsilon(F) = \{y \in X, \text{dist}(y, F) < \varepsilon\}.$$

Theorem. Assume $\phi_{n,m}$ are injective with slow dimension growth.

1 (Su, Elliott-G) \mathcal{A} is of real rank zero \iff

$\forall \mathcal{A}_n, \varepsilon > 0, \exists m$ such that for any $y, y' \in X_{m,j}$, $Sp\phi_{n,m}^{i,j}|_y$ and $Sp\phi_{n,m}^{i,j}|_{y'}$ can be paired within ε .

2. (Dadarlat-Nagy-Nemethi-Pasnicu) \mathcal{A} is simple \iff

$\forall \mathcal{A}_n, \varepsilon > 0, \exists m$ such that for any $y \in X_{m,j}$, $Sp\phi_{n,m}^{i,j}|_y$ is ε -dense in $X_{n,i}$. That is, $X_{n,i} \subseteq B_\varepsilon(\phi_{n,m}^{i,j}|_y) \forall y \in X_{m,j}$.

3. (Pasnicu) \mathcal{A} has ideal property \iff

$\forall \mathcal{A}_n, \varepsilon > 0, \exists m$ such that for any $y, y' \in X_{m,j}$,

$$Sp\phi_{n,m}^{i,j}|_{y'} \subseteq B_\varepsilon(\phi_{n,m}^{i,j}|_y).$$

Classification of real rank zero AH-algebras with slow dimension growth (Elliott-G, Dadarlat-G)

Classification of simple AH-algebras with very slow dimension growth (G, Elliott-G-Li)

How about AH-algebras with ideal property? Why do we study this class of C^* -algebras?

Three reasons:

1. It unifies real rank zero C^* -algebras and simple C^* -algebras.

2. There are some natural C^* -algebras with ideal property which do not belong to the class of real rank zero C^* -algebras and the class of simple C^* -algebras.

3. There are some C^* -algebras, for which it is easy to prove they have ideal property. But it is difficult to prove that they have real rank zero. On the other hand, if the classification theorem holds for C^* -algebras with ideal property, then it will imply the C^* -algebras have real rank zero.

Proposition [Sierakowski]. Let (\mathcal{A}, G, α) be a C^* -dynamical system, where G is a discrete amenable group and the action of G on $\hat{\mathcal{A}}$ is free, then $\mathcal{A} \rtimes_{\alpha} G$ has ideal property provided that \mathcal{A} has ideal property. In general, such \mathcal{A} is not of real rank zero and also not simple.

Ex 1. (Pasnicu) $\mathcal{A} = C(X)$, $\dim(X) = 0$, $G = \mathbb{Z}$ acts on X freely, then $C(X) \rtimes_{\alpha} \mathbb{Z}$ has ideal property. If the action is not minimal, then $C(X) \rtimes_{\alpha} \mathbb{Z}$ is not simple. Also it is not known whether $C(X) \rtimes_{\alpha} \mathbb{Z}$ is of real rank zero. But the classification for this class of C^* -algebras, implies that $C(X) \rtimes_{\alpha} \mathbb{Z}$ is of real rank zero.

Ex 2. Let $\alpha : S^3 \rightarrow S^3$ be a minimum diffeomorphism, $\beta : X \rightarrow X$ a free action which is not minimum, where X is a locally compact metric space with $\dim(X) = 0$. Then $C_0(S^3 \times X) \rtimes_{\alpha \times \beta} (\mathbb{Z} \oplus \mathbb{Z})$ has ideal property. But in general, is not of real rank zero, also not simple.

Let A be a unital C^* -algebra, $U_0(A)$ be the set of unitaries in A which can be connected to $\mathbf{1}$ by a path of unitaries. Then for any $u \in U_0(A)$, there are finitely many self adjoint elements h_1, h_2, \dots, h_n such that

$$u = e^{ih_1} e^{ih_2} \dots e^{ih_n}.$$

If every unitary $u \in U_0(A)$ can be written as a product of at most n exponentials, then we say that the exponential rank of A is at most n .

If every unitary $u \in U_0(A)$ can be approximated arbitrarily well by a product of at most n exponentials, then we say that the exponential rank of A is at most $n + \varepsilon$.

Lin (1993) Every real rank zero C^* -algebra A has exponential rank at most $1 + \varepsilon$ — $cer(A) \leq 1 + \varepsilon$

Phillips (1993) Every simple AH algebra with no dimension growth has exponential rank at most $1 + \varepsilon$ — $cer(A) \leq 1 + \varepsilon$

It leaves the following problem open (Phillips, 1993):
Whether a simple AH algebra with slow dimension growth has exponential rank at most $1 + \varepsilon$?

The classification of [Elliott-G-Li, 2007] (or Reduction Theorem of [G, 2002]) implies that the answer is yes for the case of very slow dimension growth.

In Phillips' proof of *cer*, he used the following lemma from topology.

Lemma. For any $d > 0$, there is an M with the following property: for any compact Metrizable space X , with $\dim(X) \leq d$ and any $u_0 : X \rightarrow SU_n$ (n arbitrary) such that u_0 is homotopic to constant function $u_1(x) = 1$, $\forall x \in X$, there is a homotopy path

$$v : X \times [0, 1] \rightarrow SU_n$$

with

$$v(x, 0) = u_0(x), \quad v(x, 1) = 1$$

and $\forall x \in X$, $\{v(x, t)\}_{0 \leq t \leq 1}$ has length at most M .

This lemma can be used only for the case of no dimension growth.

Lemma. (G) For any compact metric space X , let $n \geq \frac{\dim(X)+1}{2}$. If $u_0 : X \rightarrow SU_n$ is homotopic to constant function $u_1(x) = 1, \forall x \in X$, then there is a homotopy path $v : X \times [0, 1] \rightarrow SU_n$ with $v(x, 0) = u_0(x), v(x, 1) = 1$ and $\forall x \in X, \{v(x, t)\}_{0 \leq t \leq 1}$ has length at most 2π .

Theorem(G) All the AH algebras with ideal property and with slow dimension growth has exponential rank at most $1 + \varepsilon$.

(In particular, it answers the question of Phillips affirmatively: All simple AH algebras with slow dimension growth has exponential rank at most $1 + \varepsilon$.)

In the proof of the Lemma: Let

$$\Omega(SU_n) = \{f : [0, 1] \rightarrow SU_n \text{ with } f(0) = f(1) = 1\}$$

$$\Omega^{2\pi}(SU_n) =$$

$$\{f : [0, 1] \rightarrow SU_n \text{ with } f(0) = f(1) = 1 \text{ and length of } f \leq 2\pi\}.$$

Then one can use energy function on the loop space

$$\Omega(SU_n) \text{ as Morse function to prove } \pi_i(\Omega(SU_n), \Omega^{2\pi}(SU_n)) =$$

0 for $i \leq 2n$. This is similar to Bott's proof for Bott pe-

riodicity. There Bott used different metric and proved

$$\pi_i\left(\Omega(SU_n, 1, -1), \Omega^{\pi\sqrt{n}}(SU_n, 1, -1)\right) = 0 \text{ for } i \leq 2n.$$

Reduction Theorem G-Jiang-Li-Pasnicu: An AH-algebra with ideal property and with very slow dimension growth can be rewritten as AH-algebra with spaces $X_{n,i}$ being S^1 , $T_{II,k}$, $T_{III,k}$ and S^2 , where $T_{II,k}$ is a connected 2-dimensional simplicial complex with $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$, $H^1(T_{III,k}) = 0$ and $T_{III,k}$ is a connected 3-dimensional simplicial complex with $H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}$, and $H^1(T_{III,k}) = 0 = H^2(T_{III,k})$.

The case of real rank zero AH-algebras is due to Dadarlat-G.

The case of simple AH-algebras is due to G.

In the proof, we use the previous lemma to calculate

$$K_* \prod_{n=1}^{\infty} M_{k_n}(C(X_n))$$

with $k_n \geq \dim(X_n)$.

This could be used to prove that certain completely positive map from $C(X)$ to $M_{k_n}(C(X_n))$ (where $X = T_{II,k}, T_{III,k}$ or S^2) can be approximated by a homomorphism.

For the reduction theorem for the torsion free case, we do not need this calculation, because $C(S^1)$ is stably generated.

In a joint work with Jiang and Li, we will classify all AH algebras with ideal property with very slow dimension growth condition.

Invariant: Involve order on K-theory and order on $\text{mod} - p$ -K-theory. Trace spaces (of \mathcal{A} and of ideals of \mathcal{A}), and maps between trace spaces.

Previously, Stevens & Ji-Jiang classified AH algebras with ideal property for the case $X_{n,i}$ being interval.